Curves!

Implicitly defined curves

e.g. $x^2+y^2=1$

Parametrized curves

e.g. $\alpha(t)=(\cos(t),\sin(t))$

$\beta(t)=(\cos(3t),\sin(3t))$

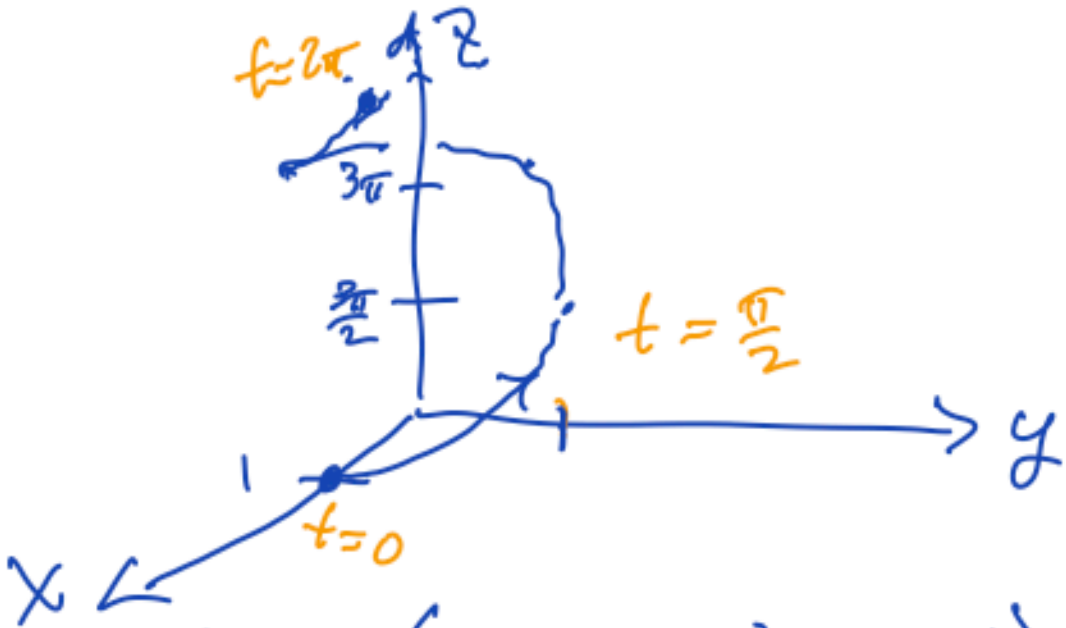
both of these are the unit circle (CCW).

$t=\frac{3\pi}{4}$ for α

$t=\frac{\pi}{4}$ for β

β is 3x as fast.

$\gamma(t)=(\cos(t),\sin(t),3t)$



$$\gamma(t) = (\cos(t), \sin(t), 3t)$$

Calculus with curves.

Defn of derivative.

Calc 1: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Same formula for curves –
we just subtract the vectors:

$$\alpha'(t) = \lim_{h \to 0} \frac{\alpha(t+h) - \alpha(t)}{h}.$$

= instantaneous rate of change of α as t increases.

= velocity of α at time t

vector

length = speed = $\|\alpha'(t)\|$

direction = direction of motion

always tangent to the curve.

Example: Helix

$$H(t) = (\cos(t), \sin(t), 3t)$$

$$H'(t) = (-\sin(t), \cos(t), 3)$$

$$H''(t) = (-\cos(t), -\sin(t), 0)$$

acceleration vector!

Examples ① $A(t) = (\cos(3t), \sin(3t)+1, 4t)$

② $B(t) = (\cos(t^3+t), \sin(t^3+t), 3)$

③ $C(t) = (e^{t^2}, 2e^{2t^2}-5)$

④ $D(t) = ((4+\cos(22t))\cos(3t), (4+\cos(22t))\sin(3t), \sin(22t))$

Instructions: Find derivative
Guess what the graph looks like.

1. $A(t) = (\cos(3t), \sin(3t)+1, 4t)$ ← helix
axis is at $x=0, y=1$.

$A'(t) = (-3\sin(3t), 3\cos(3t), 4)$

2. $B(t) = (\cos(t^3+t), \sin(t^3+t), 3)$ unit circle at height $z=3$.

$B'(t) = (-(3t^2+1)\sin(t^3+t), (3t^2+1)\cos(t^3+t), 0)$

3. $C(t) = (e^{t^2}, 2e^{2t^2}-5) = (\underset{x}{(e^{t^2})}, 2\underset{x}{(e^{t^2})}^2-5)$

Parabola

$C'(t) = (2te^{t^2}, 8te^{2t^2})$

4. $D(t) = ((4+\cos(22t))\cos(3t), (4+\cos(22t))\sin(3t), \sin(22t))$

$D'(t) = (-3\sin(3t)(4+\cos(22t)) - 22\sin(22t)\cos(3t),$
$3\cos(3t)(4+\cos(22t)) - 22\sin(22t)\sin(3t),$
$22\cos(22t))$

Examples of implicit plots – converting to parametric.

① $y^3 = 3x^2 + x$.

let $x(t) = t$

$y^3 = 3t^2 + t \Rightarrow y = \sqrt[3]{3t^2+t}$

$\alpha(t) = \left(t, \sqrt[3]{3t^2+t}\right)$.

② $x^2 + 4y^2 = 1$

We'd like to try $(\overset{x}{\cos(t)}, \overset{y}{\sin(t)})$

$\rightsquigarrow \cos^2(t) + 4\sin^2(t) = 1$ false.

Instead, use $\boxed{(x,y) = (\cos(t), \tfrac{1}{2}\sin(t))}$

$\rightsquigarrow \cos^2(t) + 4\left(\tfrac{1}{2}\sin(t)\right)^2 = \cos^2(t)$

$+ \sin^2(t) = 1.$ ✓

Recall

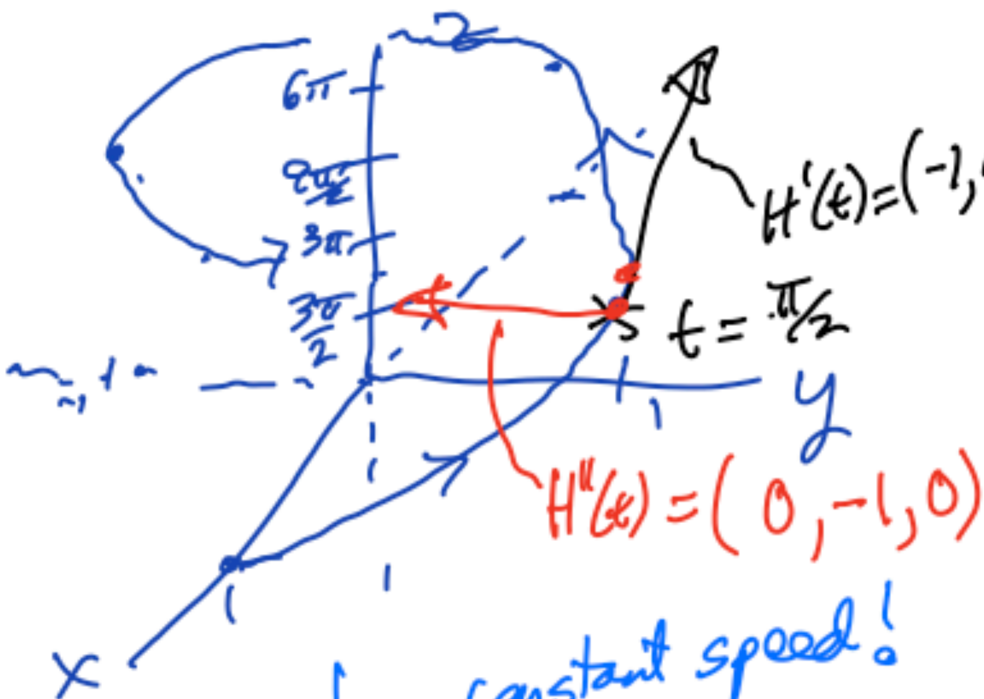
Example: Helix

$$H(t) = (\cos(t), \sin(t), 3t)$$

$$H'(t) = (-\sin(t), \cos(t), 3)$$

$$H''(t) = (-\cos(t), -\sin(t), 0)$$

acceleration vector!



This curve has constant speed!

$$\text{speed} = \|H'(t)\| = \|(-\sin(t), \cos(t), 3)\|$$

$$= \sqrt{\sin^2(t) + \cos^2(t) + 9} = \sqrt{1+9} = \boxed{\sqrt{10}}$$